TUTORIAL NOTES FOR MATH4220

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1. EXAMPLES OF FIRST ORDER PARTIAL DIFFERENTIAL EQUATIONS

Let us recall some elementary useful facts for solving the first order partial differential equations.

Theorem 1 (Lagrange theorem)**.** *Let $\Omega \subset \mathbb{R}^3$. Suppose $\varphi, \psi \in C^1(\Omega \times \mathbb{R})$ with $\nabla\varphi \times \nabla\psi \neq 0$ such that*

$$\varphi(x, y, z) = c_1, \quad \psi(x, y, z) = c_2,$$

where c_1, c_2 are two constants, form the solution of

$$\begin{aligned} \frac{dx}{ds} &= a(x, y, z), \\ \frac{dy}{ds} &= b(x, y, z), \\ \frac{dz}{ds} &= c(x, y, z), \end{aligned}$$

or equivalently,

$$\frac{dx}{a(x, y, z)} = \frac{dy}{b(x, y, z)} = \frac{dz}{c(x, y, z)}.$$

Then the implicit equation

$$G(\varphi, \psi) = 0,$$

for any $G \in C^1(\Omega \times \mathbb{R})$, is a general solution of the quasi-linear equation

$$a(x, y, z)\partial_x z + b(x, y, z)\partial_y z = c(x, y, z).$$

In the following, we discuss some examples concerning first order differential equations.

Problem 2. Find a general solution $u(x, y)$ to

$$xu\partial_x u - yu\partial_y u = x^2 - y^2.$$

Solution. Since the characteristic system gives

$$\frac{dx}{xu} = \frac{dy}{-yu} = \frac{du}{x^2 - y^2},$$

then by the Lagrange theorem, the solution is determined by

$$G(xy, x^2 + y^2 - u^2) = 0,$$

where $G \in C^1$ is an arbitrary function.

Problem 3. Solve the Cauchy problem

$$u\partial_x u + \partial_y u = 2,$$

with

$$u(x, x) = x, \quad x \neq 1.$$

Solution. In view of the given initial data, the initial curve is given by

$$x(0, s) = s, \quad y(0, s) = s, \quad u(0, s) = s.$$

In addition, the characteristic equations are given by

$$\begin{aligned} \frac{dx(t,s)}{dt} &= u(t,s), \\ \frac{dy(t,s)}{dt} &= 1, \\ \frac{du(t,s)}{dt} &= 2, \end{aligned}$$

therefore

$$x(t, s) = t^2 + ts + s, \quad y(t, s) = t + s, \quad u(t, s) = 2t + s,$$

solving t, s in terms of x, y, the solution is obtained as

$$u(x, y) = \frac{y^2 - 2y + x}{y - 1}, \quad y > 1.$$

Problem 4. Find the maximal range of $t \geq 0$ for u to be a continuous solution of the Cauchy problem

$$\partial_t u + u\partial_x u = 0,$$

with

$$u(0, x) = \begin{cases} 1, & x \leq 0, \\ 1 - x, & 0 \leq x \leq 1, \\ 0, & x \geq 1. \end{cases}$$

Solution. For arbitrary $x_0 \in \mathbb{R}$, the characteristic line $\Gamma_{x_0} : (s, x(s))$ is given by

$$\frac{dx(s)}{ds} = u(s, x(s)), \quad x(0) = x_0,$$

then by the equation,

$$\frac{du(s, x(s))}{ds} = 0,$$

which implies u is constant along Γ_{x_0}. Therefore by the initial condition, the continuous solution exists for $t \in [0, 1)$.

Moreover, we have

$$u(t, x) = \begin{cases} 1, & x \leq t, \\ \frac{1-x}{1-t}, & t \leq x \leq 1, \\ 0, & x \geq 1, \end{cases}$$

for $t \in [0, 1)$.

A Supplementary Problem

Problem. Consider

$$a(x,y)\partial_x u + b(x,y)\partial_y u + c(x,y)u = f(x,y).$$

Let $\varphi \in C^1(\Omega)$ such that $\varphi(x,y) = k$ is the family of characteristic curves for

$$a(x,y)\partial_x u + b(x,y)\partial_y u = 0,$$

where $k \in \mathbb{R}$ is the parameter. Let $\psi = \psi(x,y)$ be functionally independent from φ in Ω. Show that the change of variables

$$\xi = \varphi(x,y),$$
$$\eta = \psi(x,y),$$

transform the first order partial differential equation for u into an ordinary equation of the form

$$\partial_\eta v + P(\xi,\eta)v = Q(\xi,\eta),$$

where $v(\xi,\eta) = u(x,y)$.

For more materials, please refer to [1, 2, 3, 4].

References

[1] S. Alinhac, *Hyperbolic partial differential equations*, Universitext, Springer, Dordrecht, 2009.

[2] L. C. Evans, *Partial differential equations*, vol. 19 of Graduate Studies in Mathematics, American Mathematical Society, Providence, RI, 1998.

[3] Q. Han and F. Lin, *Elliptic partial differential equations*, vol. 1 of Courant Lecture Notes in Mathematics, New York University, Courant Institute of Mathematical Sciences, New York; American Mathematical Society, Providence, RI, 1997.

[4] W. A. Strauss, *Partial differential equations. An introduction*, John Wiley & Sons, Inc., New York, 1992.

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